Alternative theories of gravity

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Brans-Dicke Theory

Mach's Principle

Motivation of Brans-Dicke Theory: Attempt to incorporate Mach's Principle in a relativistic theory of gravitation

Mach's Principle

The inertia of any object is determined by the matter distribution in the rest of the universe.
A particle moves in an empty universe.

$$\mathbf{F} = m\mathbf{a}$$

$\mathbf{F} = 0$, but what about $\mathbf{a}$?

$\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2}$.

One cannot define $\mathbf{x}$ and hence $\mathbf{a}$!

The outcome:

$$m = 0$$

General Relativity

General Relativity → A geomtric description of gravity
Metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Einstein gravity, i.e. General Relativity is given by the Einstein-Hilbert action:

$$\mathcal{A}_{GR} = \int \left(\frac{R}{16\pi G} + L_m \right) \sqrt{-g} d^4x \tag{1}$$

A variation of this action with resepect to $g_{\mu\nu}$, the metric tensor, gives the field equations

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{2}$$

$$G_{\mu\nu} = G_{\mu\nu}(g_{\alpha\beta}, g'_{\alpha\beta}, g''_{\alpha\beta})$$

Solve this for the metric;
Write down the geodesic equations

$$\ddot{x^{\mu}} + \Gamma^{\mu}_{\alpha\beta}\dot{x^{\alpha}}\dot{x^{\beta}} = 0. \tag{3}$$

(Follows from the variational principle $\int ds = 0$.)
Get the trajectories.
Gravity $\rightarrow$ completely described by the metric.
General Relativity does not take care of Mach's principle.

Mass is measured in some units.
Can we define mass in a dimensionless manner?

$$m_p = (\frac{hc}{2\pi G})^{\frac{1}{2}} = 2.16x10^{-5}gm.$$

Express mass m in a dimensionelss form

$$\eta = \frac{m}{m_p}.$$

The variation of this can be taken care of by a variation of G !

Brans (1935-) and Dicke (1916-1997) modify the action by introducing a scalar field[1].
The action is modified as

$$\mathcal{A}_{BD} = \int \left(\frac{\phi R}{16\pi G} + \omega \frac{\phi_{,\mu}\phi^{,\mu}}{\phi} + L_m \right) \sqrt{-g}\, d^4x \tag{4}$$

Effective G is given as

$$G = \frac{G_0}{\phi} \tag{5}$$

This is a Non-minimally coupled Theory
ϕ is non-minimally couple to the geometry.

[1] C.H.Brans and R.H.Dicke, Phys. Rev, **124**, 925(1961)

The field equations are:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) + \frac{1}{\phi}(\phi_{;\mu;\nu} - g_{\mu\nu}\Box\phi) \quad (6)$$

These are obtained by a variation of the action with respect to $g_{\mu\nu}$.
A variation with respect to $\phi \rightarrow$ wave equation:

$$\Box\phi = \phi^{;\mu}{}_{;\mu} = \frac{\omega}{\phi}\phi^{,\mu}\phi_{,\mu} - \frac{\phi R}{2\omega} = \frac{T}{2\omega + 3} \quad (7)$$

With the matter conservation equation,

$$T^{\mu\nu}_{;\nu} = 0,$$

the wave equation is not independent $\rightarrow$ Follows as a consequence of Bianchi identities.
Now gravity is described by $g_{\mu\nu}$ plus the scalar field ϕ.

Some consequences:
The solutions are different $\rightarrow$ geodesics are different!
Effects:

- No effect on gravitational redshift!!
- Perihelion shift is different.
- Bending of light is different.
- A varying G, $|\frac{\dot{G}}{G}| = |\frac{\dot{\phi}}{\phi}|$

Weak field limit:

PPN parameters:

$$\alpha_{BD} = \alpha_{GR} \left(\frac{a\omega + b}{a\omega + c} \right) \tag{8}$$

For example, the light bending is given by

$$(\delta\theta)_{BD} = (\delta\theta)_{GR} \left(\frac{2\omega + 3}{2\omega + 4} \right) \tag{9}$$

Do you have black holes in BD theory?

NO !!

Demand of a non trivial $\phi \rightarrow$ event horizon (a null surface) is singular.

Consistent with NO HAIR Conjecture

Checked for both Schwarzchild[2] and Reissner-Nordstrom[3] analogues.

[2] A. Saa, JMP, 1996

[3] NB, S.Sen, PRD, 1998

Weak field:
The parameters in BD theory and GR are the same for $\omega \longrightarrow \infty$
Without the weak field approximation:
Wave equation:

$$< \phi > \approx < \phi_0 > + \frac{1}{\omega} \tag{10}$$

for $\omega \to \infty$.
BD Field equations reduces to GR !!
Proved to be conditional!! For $T \neq 0$, This is fine.
For $T = 0$,

$$< \phi > \approx < \phi_0 > + \frac{1}{\sqrt{\omega}} \tag{11}$$

for the same limit[4,5].

[4] NB, S. Sen, PRD, 1997
[5] V. Faraoni, PRD, 1999

Application in cosmology

For a spatially homogeneous and isotropic universe, Brans-Dicke field equations are:

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = \frac{8\pi G}{\phi}\rho_m + \frac{1}{2}\omega\frac{\dot{\phi}^2}{\phi^2} - 3\frac{\dot{a}\dot{\phi}}{a\phi} \tag{12}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\frac{8\pi G}{\phi}p_m - \frac{1}{2}\omega\frac{\dot{\phi}^2}{\phi^2} - 2\frac{\dot{a}\dot{\phi}}{a\phi} - \frac{\ddot{\phi}}{\phi} \tag{13}$$

Extra freedom for manipulation on the right hand side!!

Extended inflation[6,7]

A possibility for the solution of the graceful exit problem in inflationary scenario.

[6] Mathiazhagan and Johri, CQG, 1984
[7] La and Steihardt, PRL, 1989

Can give rise to an accelerated expansion even without a dark energy[8]
Both would require low values of ω.

[8] NB, D. Pavon. PRD, 2001

Local astronomy vs cosmology

Local astronomical observations

- perihelion shift
- light bending

Set a very stringent upper bound on ω. Present estimate ranges from 10^3 to 10^4.

Cosmology:
Both an "accelerating universe" and BD theory as a saviour from the graceful exit problem require

Very low values of ω !

Variation of G

Very small upper bound on the variation of G.

$$|\frac{\dot{G}}{G}|_0 < 10^{-10}$$

$$g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}$$

$$\bar{g}_{\mu\nu} = \phi g_{\mu\nu}$$

The action becomes[9]:

$$\mathcal{A}_{BD} = \int \left(\frac{R}{16\pi G} + \frac{(2\omega + 3)}{2} \frac{\phi_{,\mu}\phi^{,\mu\,2}}{\phi} + L_m \right) \sqrt{-\bar{g}}\, d^4x \tag{14}$$

The field equations:
Formally looks like a minimally coupled scalar field.
G is a constant
Price: Rest mass becomes a function of ϕ !
Principle of equivalence is lost, one cannot use geodesic equations!!

[9] R.H. Dicke, Phys. Rev., 1962

The parameter ω is a dimensionless constant.
Nordtvedt[10] proposal:

$$\omega = \omega(\phi)$$

This generalizes many a scalar-tensor theory of gravity, including Bekenstein's conformally invariant scalar-tensor theory.

[10] K. Nordtvedt, Jr,, J. Astrphys., 1970

f(R) gravity

Metric f(R) Gravity

- The generalized Einstein-Hilbert action for $f(R)$ gravity is

$$\mathcal{A} = \int \left[\frac{1}{16\pi G} f(R) + \mathcal{L}_m \right] \sqrt{-g}\, d^4x, \tag{15}$$

- A variation of this action with respect to the metric $\rightarrow$ field equations :

$$f'(R)R_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + \left[\Box f'(R) - \frac{1}{2} f(R) \right] g_{\mu\nu} = T^{(m)}_{\mu\nu}, \tag{16}$$

  where the prime indicates differentiation with respect to the Ricci scalar R. $T^{(m)}_{\mu\nu}$ represents the contribution to the energy momentum tensor from matter fields with a choice of unit as $8\pi G = 1$.

Friedmann Equations f(R) Gravity

- The spatially flat FRW metric is written as

$$ds^2 = dt^2 - a^2(t)[dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2], \tag{17}$$

where $a(t)$ is the time dependent scale factor.

- Due to the modification in basic field equations the Friedmann equations will be modified and the modified forms are written as

$$3H^2 = \frac{\rho_m + \rho_c}{f'}, \tag{18}$$

$$2\dot{H} + 3H^2 = -\frac{p_c}{f'}. \tag{19}$$

where $\left[\frac{1}{2}(f - Rf') - 3\dot{R}f''\frac{\dot{a}}{a}\right] = \rho_c$, *density like contribution of geometry and* $\left[\ddot{R}f'' + \dot{R}^2 f''' + 2\dot{R}f''\frac{\dot{a}}{a} - \frac{1}{2}(f - Rf')\right] = p_c$,*,the pressure like contribution of geometry and* ρ_m *is the matter density.*

Recent review: Sotiriou and Faraoni[11]

[11] T.P.Sotiriou and V.Faraoni, Rev Mod. Phys. **82**, 451 (2010)

Started in the early 1980's

$f(R^2) \rightarrow$ early inflation

Inverse powers $\rightarrow$late time acceleration
A smooth transition from the decelerated to accelerated expansion, driven by curvature!

Problems:

- Stable Schwarzchild limit
- Local astronomy